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STABILITY OF THIN NONSYMMETRIC PIECEWISE-CONVEX ELASTIC SHELLS

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Asymptotic values of the upper critical loads are determined for local buckling in the zone of absence of regularity of thin elastic piecewise-convex, shallow, nonsymmetric and nonshallow shells of revolution subjected to external discontinuous loads. Examples of shallow ellipsoidal and nonshallow spherical shells with a discontinuity in the meridian along the parallel are considered for uniform external pressure, and subject to a load lumped along the parallel in the case of a smooth surface.

An asymptotic method using the presence of the natural small parameter of relative thin-walledness in the shell theory equations is applied. The asymptotic values of the mentioned critical loads are determined as the least branchpoints of the nonlinear equations of the internal edge effect occuring because of absence of smoothness of the shell middle surface and the external load, by using an electronic computer.

1. Formulation of the problem. A nonlinear variant of the theory of medium bending of an elastic shallow shell with a piecewise-convex middle surface subjected to a transverse load is considered [1]:

$$\varepsilon^2\Delta^2 w - [w - z, F] = q, \quad \varepsilon^2\Delta^2 F + {}^1/_2\,[w, w] - [z, w] = 0 \tag{1.1}$$

$$\varepsilon^2 F_{xx} = \frac{1}{1-\nu^2}\left[v_y + z_{yy}w + \frac{1}{2}w_y^2 + \nu\left(u_x + z_{xx}w + \frac{1}{2}w_x^2\right)\right]$$

$$\varepsilon^2 F_{yy} = \frac{1}{1-\nu^2}\left[u_x + z_{xx}w + \frac{1}{2}w_x^2 + \nu\left(v_y + z_{yy}w + \frac{1}{2}w_y^2\right)\right]$$

$$\varepsilon^2 F_{xy} = -\frac{1}{2(1+\nu)}\left[u_y + v_x + 2z_{xy}w + w_x w_y\right]$$

$$\Delta w = w_{xx} + w_{yy}, \quad [F, w] = F_{xx}w_{yy} + F_{yy}w_{xx} - 2F_{xy}w_{xy}$$

All the quantities in (1.1) are dimensionless and connected with the dimensional relationships presented in [2]. Here z is the piecewise-convex middle surface with absence of regularity along a horizontal line Γ_1. It is assumed that the shell occupies a finite simply-connected domain D with the boundary Γ in planform. Here z_1 denotes the strictly convex part of the surface z. bounded by the curve Γ_1 and z_2 denotes a strictly convex surface one of whose edges coincides with Γ_1 and the other with the shell boundary Γ. Let us consider the curves Γ and Γ_1 to be sufficiently smooth, to have no common points, and

$$z_1(s) = z_2(s) \neq 0, \quad z_{1\rho}(s) \neq z_{2\rho}(s), \quad s \in \Gamma_1; \quad z_2(s) = 0, \quad s \in \Gamma \tag{1.2}$$

The load function $q(x, y)$ is given in the form

$$q(x, y) = q_0(x, y) + P(s)\,\delta(s), \quad s \in \Gamma_1 \tag{1.3}$$

where $q_0(x, y)$ is a sufficiently smooth function in the domain $D + \Gamma$, $\delta(s)$ is a delta-function, and $P(s)$ is the intensity of the concentrated load. Boundary conditions corresponding to a clamped or a moving hinge support of the shell edge are satisfied on the contour Γ

$$\begin{aligned} &1)\ u = v = w = w_\rho = 0, \quad s \in \Gamma \\ &2)\ F = F_\rho = w = w_{\rho\rho} + \nu(w_{ss} - \varkappa w_\rho) = 0, \quad s \in \Gamma \end{aligned} \tag{1.4}$$

The solvability and differential properties of solutions of the boundary value problems (1.1)-(1.4) have been obtained in [3], from which there follows, in particular, that the functions w and F remain continuous together with their first and second derivatives in passing through Γ_1 (continuity of the angles of rotation, the stresses and the moments)

$$\begin{aligned} &w_1 = w_2, \quad w_{1\rho} = w_{2\rho}, \quad w_{1\rho\rho} = w_{2\rho\rho} \\ &F_1 = F_2, \quad F_{1\rho} = F_{2\rho}, \quad F_{1\rho\rho} = F_{2\rho\rho}, \quad s \in \Gamma_1 \end{aligned} \tag{1.5}$$

It has been shown in [4-7] that the asymptotic values of the upper critical load for sufficiently thin smooth shells are determined by local buckling far from the edge (the principle "B" of Pogorelov) or by phenomena in the edge effect zone. Moreover, an abrupt change in the strain, moment, and force fields occurs also in the neighborhood of Γ_1 due to absence of smoothness of the middle surface and the load (internal edge effect phenomenon). This results in the fact that snapping of the shell can start in the neighbourhood of Γ_1 for loads smaller than in the case of smooth surfaces and loads.

2. Construction of the asymptotic. An asymptotic method [2,5-8] is developed here to determine the upper critical loads.

Let us note that here, as in [2,8], it is assumed that the number of azimuthal waves does not grow too rapidly along the line Γ_1 as $\varepsilon \to 0$. Limiting ourselves to the con-construction of the principal terms, let us construct the asymptotic expansions of the solutions of problems (1.1)-(1.4) as $\varepsilon^2 \to 0$ in the form

$$\begin{aligned} &w_1(x, y, \varepsilon) \sim w_{01}(x, y) + \varepsilon G_1(x, y, \varepsilon), \quad F_1(x, y, \varepsilon) \sim F_{01}(x, y) + \\ &\qquad \varepsilon H_1(x, y, \varepsilon) \\ &w_2(x, y, \varepsilon) \sim w_{02}(x, y) + \varepsilon G_2(x, y, \varepsilon) + \varepsilon g_0(x, y, \varepsilon) \\ &F_2(x, y, \varepsilon) \sim F_{02}(x, y) + \varepsilon H_2(x, y, \varepsilon) + \varepsilon h_0(x, y, \varepsilon) \end{aligned} \tag{2.1}$$

Here the subscript $i = 1$ if $(x, y, z) \in z_1$, and $i = 2$ if $(x, y, z) \in z_2$. The functions w_{0i}, F_{0i} correspond to the membrane form of shell equilibrium coincident with the initial surface and are determined to form the system (1.1) for $\varepsilon^2 = 0$

$$w_{0i} = 0 \quad [z_i, F_{0i}] = q \tag{2.2}$$

with boundary conditions corresponding to (1.4)

$$\begin{aligned} &1)\ [F_{02,\rho\rho} - \nu F_{02,ss} + \varkappa_0 \nu F_{02,\rho}]_\Gamma = w_{02}(s) = 0, \quad s \in \Gamma \\ &2)\ F_{02}(s) = w_{02}(s) = 0, \quad s \in \Gamma \end{aligned} \tag{2.3}$$

The functions of boundary-layer type g_0, h_0 are constructed in [2]. They are concentrated in the neighborhood of the shell edge Γ and cancel the residuals in the satisfaction of the boundary conditions (1.4) for w_{02}, F_{02}. Because of the strict convexity of the surfaces z_1 and z_2 it follows that the equations in (2.2) are elliptic. It can then be shown that

$$F_{01} = F_{02}, \quad F_{01,\rho} \neq F_{02,\rho} \tag{2.4}$$

if $P(s) \neq 0$ or $z_{1\rho}(s) \neq z_{2\rho}(s)$. Comparing (1.5) and (2.4), we find that the solutions of problems (2.2), (2.3) have worse differential properties than the solutions of problems (1.1)-(1.4). The functions of boundary layer type G_i, H_i cancel the discontinuities in the derivatives, improving the differential properties in the solutions w_{0i}, F_{0i}, i.e., the functions $w_{0i} + \varepsilon G_i$, $F_{0i} + \varepsilon H_i$ have differential properties satisfying conditions (1.5) in the neighborhood of Γ_1. Therefore, the internal boundary layer phenomenon holds [9-15]. However, in contrast to the papers mentioned, here bifurcation holds in the edge effect zone.

The functions G_i, H_i are concentrated in the neighborhood of the line Γ_1 and are determined from the internal edge effect equations. To derive these equations, let us go over to the local coordinates (ρ, φ) in the neighborhood, and by using (2.1) let us carry out a construction [2] related to stretching the boundary layer on both sides of Γ_1. Consequently, we obtain a system to determine G_i, H_i $(i = 1, 2)$ which with the aid of the change of variables ($\varkappa(s)$ is the curvature of the contour Γ_1)

$$\begin{aligned} &\frac{\partial H_i(t_i)}{\partial t_i} = (-1)^i \alpha_i, \quad \frac{\partial G_i(t_i)}{\partial t_i} = (-1)^i \beta_i, \quad t_1 = \frac{\rho}{\varepsilon} > 0 \\ &t_2 = -\frac{\rho}{\varepsilon} < 0 \\ &Q_i = 2 f_i c_i^{-1}, \quad \tau_i = t_i (\varkappa c_i)^{1/2}, \quad \varkappa > 0 \\ &c_i = -\varkappa^{-1} (z_{i,xx} \rho_y^2 + z_{i,yy} \rho_x^2 - 2 z_{i,xy} \rho_x \rho_y) \\ &f_i = -\varkappa^{-1} (F_{0i,xx} \rho_y^2 + F_{0i,yy} \rho_x^2 - 2 F_{0i,xy} \rho_x \rho_y) \end{aligned} \tag{2.5}$$

reduces to a system of nonlinear ordinary differential equations

$$c_i \frac{\partial^2 \alpha_i}{\partial \tau_i^2} + \frac{1}{2} \beta_i^2 + c_i \beta_i = 0, \quad c_i \frac{\partial^2 \beta_i}{\partial \tau_i^2} + \frac{1}{2} Q_i c_i \beta_i - \alpha_i \beta_i - c_i \alpha_i = 0 \tag{2.6}$$

Substituting (2.1) into (1.5), taking account of (2.4), and again applying stretching of the boundary layer, we obtain the conjugate conditions at $\tau_1 = \tau_2 = 0$ for the system (2.6) [8]:

(2.7)

$$1^\circ.\ \alpha_1(0) - \alpha_2(0) = F_{01,\rho}(0) - F_{02,\rho}(0)$$

$$2^\circ.\quad c_1^{1/2}\frac{\partial\alpha_1(0)}{\partial\tau_1} = -c_2^{1/2}\frac{\partial\alpha_2(0)}{\partial\tau_2}$$

$$3^\circ.\ \beta_1(0) = \beta_2(0)$$

$$4^\circ.\quad c_1^{1/2}\frac{\partial\beta_1(0)}{\partial\tau_1} = -c_2^{1/2}\frac{\partial\beta_2(0)}{\partial\tau_2}$$

Moreover, the conditions

$$\alpha_i(\infty) = \beta_i(\infty) = 0 \quad (i = 1, 2) \tag{2.8}$$

result from the requirement that the boundary layer functions decrease at infinity. Therefore, for arbitrarily given z and q satisfying conditions (1.2) and (1.4), the solution of the equations for the principal term in the internal edge effect zone reduces to the identical system (2.6)-(2.8) in which the dependence of the critical load on the shape of the middle surface and the loading method is taken into account by the parameters c_i, Q_i. It is interesting that the system (2.6)-(2.8) agrees with the internal edge effect system for a shallow shell of revolution with a piecewise-smooth meridian.

Furthermore, let us introduce the quantity

$$\sigma = \max_s Q_2 = \max_s (2f_2c_2^{-1}), \quad s \in \Gamma_1 \tag{2.9}$$

as the loading parameter.

According to [5], the asymptotic value of the upper critical load for local buckling in the neighborhood of Γ_1 is determined by the least branchpoint σ^* of the problem (2.6)-(2.8). Let us seek the solution of the problem (2.6)-(2.8) in the form

(2.10)

$$\alpha_k(\tau_k) = \sum_{m+n\geqslant 1} \gamma_{kmn}^{(1)} x_k^m y_k^n, \qquad \beta_k(\tau_k) = \sum_{m+n\geqslant 1} \gamma_{kmn}^{(2)} x_k^m y_k^n$$

$$x_k = x_{0k}\exp(-r_k\tau_k), \quad y_k = y_{0k}\exp(-p_k\tau_k)$$

$$r_k = -a_k - ib_k, \quad p_k = -a_k + ib_k, \quad a_k = \left(\frac{1}{2} - \frac{Q_k}{8}\right)^{1/2},$$

$$b_k\left(\frac{1}{2} + \frac{Q_k}{8}\right)^{1/2}$$

All the constructions are evidently valid for $Q_k < 4$.

Substituting (2.10) into (2.6) and collecting terms in idenitcal powers of x_k, y_k, we determine the coefficients $\gamma_{kmn}^{(i)}$. Then substituting (2.10) into (2.7), we obtain a system of nonlinear algebraic equations to determine x_{0k}, y_{0k}

$$G_1(\gamma^{(1)}) \equiv \sum_{m+n\geqslant 1} (\gamma_{1mn}^{(1)} x_{01}^m y_{01}^n - \gamma_{2mn}^{(1)} x_{02}^m y_{02}^n) = \frac{1}{2}(Q_1 c_1 - Q_2 c_2)$$

$$G_2(\gamma^{(1)}) \equiv \sum_{m+n\geqslant 1} [c_1^{1/2}(mr_1 + np_1)\gamma_{1mn}^{(1)} x_{01}^m y_{01}^n + \qquad (2.11)$$

$$c_2^{1/2}(mr_2 + np_2)\gamma_{2mn}^{(1)} x_{02}^m y_{02}^n] = 0$$

$$G_1(\gamma^{(2)}) = 0, \quad G_2(\gamma^{(2)}) = 0$$

We use the method elucidated in [6] to solve the system (2.11), (2.12) on an electronic computer. The condition that its Jacobian vanishes is used to determine the branch points of the system (2.11). The computations are checked by using the first integrals of the system (2.6).

Passing to dimensional variables, we obtain a formula to determine the asymptotic values of the critical load for local buckling in the neighborhood of the line Γ_1

$$p^* \approx \frac{E}{\sqrt{12(1-\nu^2)}} \frac{h^2}{a^2} \sigma^*(c_i, Q_i) \qquad (2.12)$$

When $F_{01,ss} = F_{02,ss} = 0$ for $s \in \Gamma_1$, the quantity σ^* depends only on the two parameters c_1/c_2, Q_1/Q_2.

3. Examples for shallow shells. 1°. Ellipsoidal shell with piecewise-convex surface subjected to a uniform external load for a mobile hinge-supported edge. The equations of the shell middle surfaces and the contours Γ and Γ_1 are given in the form

$$z_1 = M - 1/2(cx^2 + by^2), \quad z_2 = 1 - 1/2(k_1x^2 + k_2y^2), \quad z_2|_\Gamma = 0 \qquad (3.1)$$

$$X = \left(\frac{2}{k_1}\right)^{1/2} \cos\varphi, \quad Y = \left(\frac{2}{k_2}\right)^{1/2} \sin\varphi, \quad X_1 = \lambda X, \quad Y_1 = \lambda Y, \quad k_i > 0$$

There follows from the condition $z_1(\Gamma_1) = z_2(\Gamma_1) = 0$ that

$$c = \lambda^{-2}k_1(M - 1 + \lambda^2), \quad b = \lambda^{-2}k_2(M - 1 + \lambda^2) \qquad (3.2)$$

The functions F_{01}, F_{02} are determined from (2.2) and we have the form

$$F_{01} = \frac{q}{2k_1k_2}\left(1 - \lambda^2 + \frac{k_1\lambda^2}{c}\right) - \frac{q}{4cb}(cx^2 + by^2) \qquad (3.3)$$

$$E_{02} = \frac{q}{4k_1k_2}(2 - k_1x^2 - k_2y^2)$$

By using (3.1)-(3.3), we obtain from (2.5)

$$\frac{c_1}{c_2} = \frac{c}{k_1} = \frac{b}{k_2}, \quad Q_1 = \frac{q}{cb}, \quad Q_2 = \frac{q}{k_1k_2}, \qquad (3.4)$$

$$[F_{01,\rho} - F_{02,\rho}]_{\rho=0} = \frac{Q_1c_1 - Q_2c_2}{2}$$

The asymptotic values of the critical load for local buckling in the neighborhood of the line Γ_1 in dimensional variables have the form

$$p^* = \frac{E\sigma^*}{\sqrt{12(1-\nu^2)}} \frac{h^2}{R_1 R_2} \qquad (3.5)$$

It follows from (3.4) that $F_{01,ss} = F_{02,ss} = 0$ for $s \in \Gamma_1$, $Q_1 / Q_2 = (c_2 / c_1)^2$, and the quantity σ^* depends only on the parameter c_1 / c_2. Certain values of $\sigma^* = \sigma^*(c_1/c_2)$ are presented below. For c_1/c_2 equal to 1.005, 1.004, 1.094, 1.122, 1.316, 1.438, 1.669 the quantities σ^* equal, respectively, 3.968, 3.734, 3.449, 3.303, 2.562, 2.248, 1.827.

2°. Ellipsoidal shell with a smooth surface subjected to a load of intensity P concentrated along the contour Γ_1 for a movable hinge-supported shell edge. The equations for Γ and Γ_1 are written in (3.1) and the equations for the middle surface and the load have the form

$$z = 1 - {}^1/_2 \; (k_1 x^2 + k_2 y^2), \quad q = P\delta(s), \quad s \in \Gamma_1 \qquad (3.6)$$

The functions F_{0i} are determined from the problem (2.2) and 2) from (2.3) and (2.4)

$$[z, F_{0i}] \equiv -k_1 F_{0i,yy} - k_2 F_{0i,xx} = P\delta(s), \quad F_{02}|_\Gamma = 0 \qquad (3.7)$$

It can be shown that

$$F_{01} = 2A \ln \lambda, \quad F_{02} = A \ln \frac{k_1 x^2 + k_2 y^2}{2}, \quad Q_1 = 0, \quad Q_2 = \frac{2A}{\lambda^2} \qquad (3.8)$$

$$A = P\lambda \sqrt{2 \max(k_1, k_2)} \frac{E(l)}{\pi k_1 k_2}, \quad l^2 = 1 - \min\left(\frac{k_1}{k_2}, \; \frac{k_2}{k_1}\right)$$

Here $E(l)$ is the complete normal Legendre elliptic integral of the second kind [16]. The asymptotic value of the critical load in dimensional variables has the form

$$p^* = \frac{1.309E}{\sqrt{12(1-\nu^2)}} \left[\frac{a\lambda\pi}{2\sqrt{2\max(k_1, k_2)}\, E(l)}\right] \frac{h^2}{R^2} \qquad (3.9)$$

3°. Ellipsoidal shell with piecewise-convex surface clamped along the edge under a uniform external load. In this case the middle surface and the equations of the contours Γ and Γ_1 are given by (3.1). The functions F_{0i} are determined from (2.2) under boundary condition 1) from (2.3) and (2.4) and have the form

$$F_{01} = c_0 + q\{[Ka(k_1^2 - k_2^2) + k_1]x^2 + [k_2 - Kb(k_1^2 - k_2^2)]y^2\}[2(ak_2 + bk_1)]^{-1} \qquad (3.10)$$

$$F_{02} = -qK\left\{(k_2 + \nu k_1)\frac{x^2}{2} + (k_1 + \nu k_2)\frac{y^2}{2}\right\} + Ax + By + D,$$

$$K = (k_1^2 + k_2^2 + 2\nu k_1 k_2)^{-1}$$

The values of the coefficients A, B, D, c_0 are not essential. The coefficients c_i, Q_i in the system (2.6) are determined from (3.10) by using (2.5) and formulas (2.6) from [2]. It follows from (3.10) that $F_{0i,ss} \not\equiv 0$ for $s \in \Gamma_1$. The asymptotic value of the critical load p in dimensional variables has the form

$$p = \frac{E\sigma^*}{\sqrt{12(1-\nu^2)}} \frac{h^2}{R_1 R_2}$$

The quantity σ^* is determined as the least branchpoint of the problem (2.6)-(2.8) for each fixed φ

4. Thin orthotropic non-shallow elastic shell of revolution. Let us apply the asymptotic method to determine the critical loads for the Reissner equations [17, 18] of axisymmetric deformations of thin non-shallow shells of revolution with piecewise-smooth meridian

$$\sigma_i \varepsilon^2 \left\{ [r_i(\Phi_i - \varphi_i)']' - \frac{\beta_i^2}{r_i} \cos\Phi_i (\sin\Phi_i - \sin\varphi_i) + \nu_i \varphi_i' (\cos\Phi_i - \cos\varphi_i) \right\} = \beta_i^2 (\Psi_i \sin\Phi_i - T_i \cos\Phi_i), \quad T_i = -\int_{s_i}^{s} r_i q_i \, ds \tag{4.1}$$

$$\mu_i \varepsilon^2 \left\{ (r_i \Psi_i')' - \frac{\beta_i^2}{r_i} \Psi_i \right\} = \cos\Phi_i - \cos\varphi_i - \mu_i \varepsilon^2 \left\{ (r_i^2 p_i)' + \left(\beta_i^2 \frac{\sin\Phi_i}{r_i} + \nu_i \Phi_i'\right)(\Psi_i \sin\Phi_i - T_i \cos\Phi_i) + \nu_i r_i (p_i \cos\Phi_i + q_i \sin\Phi_i) \right\}$$

All the quantities in (4.1) are dimensionless and related to the dimensional quantities by means of the formulas (4.2)

$$\varepsilon^2 = \left(\frac{h_1 h_2}{a^2 \gamma^2}\right)^{1/2}, \quad \gamma = (\gamma_1 \gamma_2)^{1/2}, \quad \gamma_i^2 = 12(1 - \nu_{i\theta}\nu_{is}), \quad \beta_i^2 = \frac{E_{i\theta}}{E_{is}} = \frac{\nu_{i\theta}}{\nu_{is}}$$

$$\nu_i = \nu_{i\theta}, \quad r_{0i} = r_i a, \quad z_{0i} = z_i a, \quad \operatorname{tg}\varphi_i = \frac{z_i'}{r_i'}, \quad r_1(0) = r_2(0),$$

$$z_1(0) = z_2(0) = 0$$

$$\varphi_1(0) \neq \varphi_2(0), \quad \mu_1^2 = (C_2 C_1^{-1}), \quad \mu_2 = \mu_1^{-1}, \quad \sigma_1^2 = D_1 D_2^{-1},$$

$$\sigma_2 = \sigma_1^{-1}$$

$$C_i = E_{i\theta} h_i, \quad D_i = E_{i\theta} h_i^3 \gamma_i^{-2}, \quad E^2 = E_{1\theta} E_{2\theta}, \quad (\;)' \equiv \frac{d}{ds}(\;)$$

$$\{\Psi_{Hi}, \Psi_{Vi}, p_{Hi}, p_{Vi}\} = E\gamma\varepsilon^4 \{a^2 \Psi_i, a^2 T_i, p_i, q_i\}$$

$$M_i = \frac{a M_{si}}{D_i} = (\Phi_i - \varphi_i)' + \nu_i \frac{\sin\Phi_i - \sin\varphi_i}{r_i}$$

$$u_i = \frac{u_{0i} C_i}{a^2 \gamma E \varepsilon^2} = r_i \Psi_i' + r_i^2 p_i - \nu_i \{\Psi_i \cos\Phi_i - T_i \sin\Phi_i\}$$

Here $z_{0i}(s)$, $r_{0i}(s)$ are parametric equations of the middle surface, where s is the arclength of the meridian measured from the discontinuity point $s = 0$); $i = 1$ for $s_1 \leqslant s < 0$ and $i = 2$ for $0 < s \leqslant s_2$; $E_s E_\theta$, ν_s, ν_θ are the Young's moduli and Poisson's ratios, respectively, in the meridian and circumferential directions; a_1 is the characteristic dimension, and h is the shell thickness; ε^2 is the relative thin-walledness parameter, and β^2 is the coefficient of orthotropy. The remaining notation is analogous to that taken in [6, 18, 19].

It is assumed that the shell middle surface is sufficiently smooth and strictly convex in the intervals $s_1 \leqslant s < 0$ and $0 < s \leqslant s_2$ while the load has the form

$$p(\Phi(s), s) = p_0(\Phi(s), s) + p_1\delta(s), \quad q(\Phi(s), s) = q_0(\Phi(s), s) + q_1\delta(s) \tag{4.3}$$

The bending moment, stress, and horizontal displacement remain continuous at the discontinuity point $0 = s$ and the angle between the shell elements does not change during deformation, i.e.,

$$\sigma_1 M_1(0) = \sigma_2 M_2(0), \quad \Psi_1(0) = \Psi_2(0)$$
$$\mu_1 u_1(0) = \mu_2 u_2(0), \quad \Phi_1(0) - \varphi_1(0) = \Phi_2(0) - \varphi_2(0) \tag{4.4}$$

Finally, boundary conditions corresponding to the method of fixing the shell edge [6] should be given for $s = s_1$ and $s = s_2$.

In the case of a closed shell we have

$$\Phi_i(s_i) = \Psi_i(s_i) = \varphi_i(s_i) = r_i(s_i) = 0, \quad i = 1, 2 \tag{4.5}$$

Limiting ourselves to the construction of smooth terms of the asymptotic, we construct the solution of the problem (4.1)-(4.4) as

$$\Phi_i(s, \varepsilon) \sim \Phi_{i0}(s) + G_{i0}(t_i) + g_{i0}^{(1)}\left(\frac{s - s_1}{\varepsilon}\right) + g_{i0}^{(2)}\left(\frac{s_2 - s}{\varepsilon}\right), \quad t_1 = -\frac{s}{\varepsilon} > 0$$
$$\Psi_i(s, \varepsilon) \sim \Psi_{i0}(s) + H_{i0}(t_i) + h_{i0}^{(1)}\left(\frac{s - s_1}{\varepsilon}\right) + h_{i0}^{(2)}\left(\frac{s_2 - s}{\varepsilon}\right), \quad t_2 = \frac{s}{\varepsilon} > 0 \tag{4.6}$$

The functions $\Phi_{i0}(s)$, $\Psi_{i0}(s)$ are solutions of (4.1) to the accuracy of quantities of ε^2 order and correspond to the membrane stress mode of equilibrium. They are determined from (4.1) for $\varepsilon = 0$ and are

$$\Phi_{i0}(s) = \varphi_i(s), \quad \varphi_1(0) \neq \varphi_2(0), \quad \Psi_{10}(0) \neq \Psi_{20}(0) \tag{4.7}$$

$$\Psi_{i0}(s) = -\operatorname{ctg}\varphi_i(s)\int_{s_i}^{s} r_i(\xi)\, q_i(\varphi_i(\xi), \xi)\, d\xi, \quad i = 1, 2$$

Therefore, the internal edge effect phenomenon holds and the boundary layer functions $G_{i0}(t_i)$, $H_{i0}(t_i)$ concentrated in the neighborhood of the parallel $s = 0$, cancels the residuals in satisfying conditions (4.4) for Φ_{i0}, Ψ_{i0}.
The boundary layer functions $g_{i0}^{(k)}$, $h_{i0}^{(k)}$ $(i, k = 1, 2)$ are lumped in the neighborhood of the shell edge $(s = s_1, s = s_2)$ and are constructed in [6, 19] for different boundary condition cases. These functions should be omitted in the expansiond (4.6) upon compliance with the conditions (4.5).
Let us deduce a system of internal edge effect equations on a half-bounded line to determine $G_{i0}(t_i)$, $H_{i0}(t_i)$ By using the change of variables

$$\varphi_i^* = |\Phi_{i0}(0)|, \quad 0 < \varphi_i^* < \pi, \quad G_{i0} = G_i \operatorname{sign} \Phi_{i0}(0), \quad H_{i0} = \left(\frac{\sigma_i}{\mu_i}\right)^{1/2} \frac{H_i}{\beta_i} \tag{4.8}$$

$$t_i = \tau_i \left[\frac{r_i(0)(\sigma_i \mu_i)^{1/2}}{\beta_i \sin \varphi_i^*}\right]^{1/2}, \quad T_{i0}(0, \varkappa) = -\frac{Q_i \sin^2 \varphi_i^*}{2\beta_i} \left(\frac{\sigma_i}{\mu_i}\right)^{1/2} \operatorname{sign} \Phi_{i0}(0)$$

$$\xi_2 = \frac{\beta_2 \sigma_1}{\beta_1 \mu_1}, \quad \xi_4 = \operatorname{sign} \frac{\Phi_{10}(0)}{\Phi_{20}(0)}, \quad \xi_3 = \left(\frac{\sin \varphi_1^*}{\sin \varphi_2^*} \xi_2\right)^{1/2} \mu_1,$$

$$\xi_1 = \xi_3 \xi_4 \frac{\beta_1 \sigma_1}{\beta_2 \mu_1}$$

These equations always reduce to the same system

$$\sin \varphi_i^* \frac{d^2 G_i}{d\tau_i^2} + \frac{1}{2} Q_i^* \sin \varphi_i^* \sin G_i - H_i \sin (G_i + \varphi_i^*) = 0 \tag{4.9}$$

$$\sin \varphi_i^* \frac{d^2 H_i}{d\tau_i^2} + \cos \varphi_i^* - \cos (G_i + \varphi_i^*) = 0$$

with conditions for $\tau_i = 0$, resulting from (4.4) and also conditions that G_i, H_i decrease at infinity

$$\xi_1 \frac{dG_1(0)}{d\tau_1} + \frac{dG_2(0)}{d\tau_2} = 0, \quad \xi_2 \left[H_1(0) - \frac{1}{4} Q_1 \sin 2\varphi_1^*\right] - H_2(0) + \frac{1}{4} Q_2 \sin 2\varphi_2^* = r_1 p_1 \tag{4.10}$$

$$\xi_3 \frac{dH_1(0)}{d\tau_1} + \frac{dH_2(0)}{d\tau_2} = 0, \quad \xi_4 G_1(0) - G_2(0) = 0$$

$$G_i(\infty) = H_i(\infty) = 0, \quad i = 1, 2$$

It is convenient to set

$$\varkappa = Q_2 \neq 0, \quad \varkappa = \varkappa(\varphi_1^*, \varphi_2^*, l, p_1), \quad l = \frac{Q_1}{Q_2} \tag{4.11}$$

as the load parameter $\varkappa$.
The method of solving the problem (4.9)-(4.10) is described in Sect. 2.

5. Examples for nonshallow shells. 1° Closed isotropic shell subjected to a hydrostatic load. In this case we have

$$q_i = Q\cos\Phi_i(s), \quad p_i = -Q\sin\Phi_i(s), \quad \sigma_i = \mu_i = \beta_i = 1$$
$$\varphi_1^* = b_1, \quad \varphi_2^* = \pi - b_2, \quad \cos\varphi_i(s) = r_i'(s), \quad \xi_2 = \xi_4 = 1$$
$$T_{i0}(s) = -\frac{1}{2}Qr_i^2(s), \quad Q_i = \frac{Qr_i^2(0)}{\sin^2 b_i}, \quad \xi_1 = \xi_3 = \left(\frac{\sin b_1}{\sin b_2}\right)^{1/2} \tag{5.1}$$

Here Q is the intensity of the external pressure, b_1, b_2 are angles the shell element makes with the horizontal axis at the discontinuity point of the meridian. Passing to dimensional variables, we obtain for the asymptotic value of the critical pressure Q^*

$$Q^* = \frac{Eh^2}{\sqrt{12(1-\nu^2)}}\frac{\sin b_1 \sin b_2}{r_{10}^2(0)}\varkappa^*(b_1, b_2) \tag{5.2}$$

The values of $\eta = \varkappa^* \sin b_1 (\sin b_2)^{-1}$ are here presented in Fig. 1.

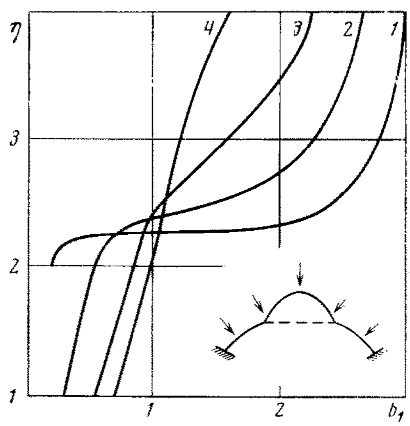

Fig. 1

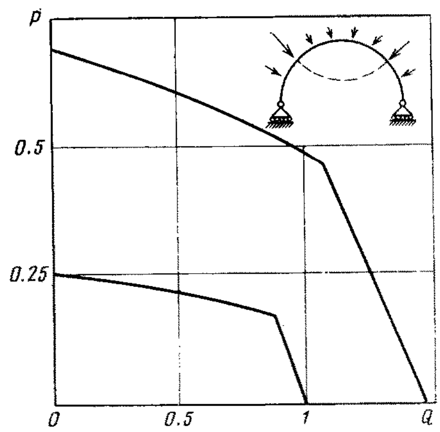

Fig. 2

Numbers 1-4 refer to the values $b_2 = 0.2, 0.5, 0.9, 1.571$ respectively. In the case of a shell consisting of spherical segments of radius aR_1 and aR_2 we have

$$\varphi_1(s) = b_1 + \frac{s}{R_1}, \quad \varphi_2(s) = \pi - b_2 + \frac{s}{R_2}, \quad r_{i0}(s) = aR_i \sin\varphi_i(s)$$
$$z_{10}(s) = aR_1[\cos b_1 - \cos\varphi_1(s)], \quad z_{20}(s) = -aR_2[\cos b_2 - \cos\varphi_2(s)], \quad r_{10}(0) = r_{20}(0) \tag{5.3}$$

We then deduce from (5.2)

$$Q^* = \frac{E\varkappa^*(b_1, b_2)}{\sqrt{12(1-\nu^2)}}\frac{h^2}{a^2R_1R_2} \tag{5.4}$$

2°. Spherical shell subjected to hydrostatic pressure Q and a force P distributed uniformly along a ring. In this case the shell middle surface and the load are given in the form

$$r_{i0}(s) = a \sin \varphi_i(s), \quad \varphi_i(s) = b + s, \quad \sigma_i = \mu_i = \beta_i = 1$$
$$p = -Q \sin \Phi(s) - P \sin \Phi(0)\,\delta(s), \quad q = Q \cos \Phi(s) + P \cos \Phi(0)\,\delta(s)$$

where b is the slope of the shell element to the abscissa axis at the point of application of the concentrated force P. We obtain from (4.7), (4.8) and the last equation in (4.1)

$$T_{10}(s) = -{}^1/_2\, Q \sin^2 \varphi_1(s), \quad \Psi_{10}(s) = -{}^1/_4\, Q \sin 2\varphi_1(s), \quad s < 0$$
$$T_{20}(s) = -{}^1/_2\, Q \sin^2 \varphi_2(s) - P \sin b, \quad \Psi_{20}(s) = -{}^1/_4\, Q \sin 2\varphi_2(s) - P \cos b, \quad s > 0$$
$$\Psi_{10}(0) \neq \Psi_{20}(0), \quad Q_1 = Q, \quad Q_2 = Q + 2P/\sin b, \quad b \neq 0$$

Solving the problem (4.1)-(4.4), we obtain neutral curves corresponding to those values of P and Q for which local buckling occurs in the neighborhood of the line of application of the concentrated load. The critical value of the vertical component of the stress function T^* at the edge of a spherical shell subjected to a hydrostatic load is determined at buckling by the formula

$$T^*(b_0) = -{}^1/_2\, \sigma^*(b_0) \sin^2 b_0$$

where b_0 is the shell aperture angle, and σ^* is the value of the upper critical buckling load which is determined by the method of fixing the edge and is presented in [6, 19, 20]. In the case under consideration, the critical value of the vertical component of the stress function equals $T_2^*(b_0 - b)$ and the neutral buckling curve is determined by the equation

$$T^*(b_0) = T_2^*(b_0 - b) \quad \text{or} \quad Q = \sigma^*(b_0) - \frac{2P \sin b}{\sin^2 b_0}$$

The general neutral curves should consist of sections corresponding to local buckling near the edge and sections corresponding to local in the neighborhood of the line of application of the load P. For certain values of b and b_0 in the case of a movable hinge fixed edge, these curves are presented in Fig. 2. It is seen from Fig. 2 that buckling starts in the neighborhood of the line of application of the concentrated load for $b_0 = 1.38$ and $b = 0.5$ (curve 1) with $Q = 1.00$ and $P = 0.486$, and near the edge with $Q = 1.25$ and $P = 0.252$. Curve 2 corresponds to the case $b = 0.2$, $b_0 = 0.8$.

3°. Influence of edge fixing on the magnitude of the upper critical load for a shell in the shape of spherical segments. Using the results in [6], let us write the asymptotic value of the upper critical load for local buckling near a shell edge in dimensional form

$$Q_1^* = \frac{E\sigma^*(b)}{\gamma} \frac{h^2}{(aR_2)^2}$$

Here aR_2 is the radius of a segment whose edge is fixed, b is the segment aperture angle at the edge, $\sigma^*(b)$ depends on the method of fixing the edge and is presented in [7, 19, 20]. By using (5.4) we write the ratio between the asymptotic values of the critical loads for local buckling near the shell edge and in the neighborhood of the broken line

$$\rho = \frac{Q_1^*}{Q_2^*} = \frac{\sigma^*(b)}{\varkappa^* \sin b_1 (\sin b_2)^{-1}} = \frac{\sigma^*(b)}{\eta}$$

Values of the quantity $\eta = \varkappa^* \sin b_1 (\sin b_2)^{-1}$ are presented for convenience in Fig. 1. For $\rho > 1$ buckling starts in the neighborhood of the discontinuity line and for $\rho < 1 - \text{в}$ in the neighborhood of the shell edge. Let us note that in the case of a clamped edge or fixed-hinge support $\sigma^*(b) = 4$ (see [4, 20] and $\rho > 1$ always).

REFERENCES

1. Mushtari, Kh.M. and Galimov, K.Z., Nonlinear Theory of Elastic Shell. Tatknigoizdat, Kazan, 1957.
2. Srubshchik, L.S., On the loss of stability of nonsymmetric, strictly convex thin shallow shells. PMM, Vol.37, № 1, 1973.
3. Vorovich, I.I., On the existence of solution in nonlinear shell theory. Dokl. Akad.Nauk SSSR, Vol.117, № 2, 1957.
4. Pogorelov, A.V., Geometric Methods in the Nonlinear Theory of Elastic Shells. "Nauka", Moscow, 1967.
5. Srubshchik, L.S., Asymptotic method of determining the critical buckling loads of shallow, strictly convex shells of revolution. PMM, Vol.36, № 4, 1972.
6. Zhukov, M.Iu. and Srubshchik, L.S., On the asymptotic value of the upper critical pressure of nonshallow spherical shells. PMM, Vol.38, № 4, 1974.
7. Zhukov, M.Iu. and Srubshchik, L.S., On the question of the buckling of the membrane state of stress of shells. Transactions, Ninth All-Union Conference on the Theory of Shells and Plates, pp.121-3, Sudostroenie, Leningrad, 1975.
8. Vorovich, I.I. and Srubshchik, L.S., Asymptotic analysis of the general equations of nonlinear shallow shell theory. Transactions, Seventh All-Union Conference on the Theory of Shells and Plates, pp. 156-9, "Nauka", Moscow, 1970.
9. Haber, S. and Levinson, N., A boundary value problem for a singular perturbed differential equation. Proc. Amer.Math.Soc., Vol.6, 1955, pp.866-72.
10. Vishik, M.I. and Liusternik, L.A., Regular degeneration and boundary layer for linear differential equations with a small parameter. Usp. Matem. Nauk, Vol.12, № 5, 1957.
11. Isakova, E.K., Asymptotics of the solution of a second order partial differential equation of parabolic type with a small parameter in the highest derivative. Dokl. Akad. Nauk SSSR, Vol.117, № 6, 1957.
12. Vasil'eva, A.B., Uniform approximation to the solution of a system differential equations with a small parameter in the derivative and application to boundary value problems. Dokl.Akad. Nauk SSSR, Vol.124, № 3, 1959.

13. Tupchiev, V. A., On angular solutions of boundary value problems with a small parameter in the derivatives in a system of first order equations. Vestnik Moscow Univ., Ser. Matem., Mekhan., № 3, 1963.
14. Srubshchik, L. S. and Iudovich, V. I., Asymptotics of weak fluid flow discontinuities with vanishing viscosity. Dokl. Akad. Nauk SSSR, Vol. 199, № 3, 1971.
15. Batishchev, V. A. and Srubshchik, L. S., Diffusion of a spherical Hill vortex with vanishing viscosity. Dokl. Akad. Nauk SSSR, Vol. 197, № 5, 1971.
16. Korn, G. and Korn, T., Handbook on Mathematics for Scientific Workers and Engineers. "Nauka", Moscow, 1974, p. 767.
17. Reissner, E., On axisymmetrical deformation of thin shells of revolution. Proc. Sympos. Appl. Math., Vol. 3, 1950, pp. 27-52.
18. Srubshchik, L. S., Asymptotics of the Reissner equation in the nonlinear theory of symmetrically loaded shells of revolution. Dokl. Akad. Nauk SSSR, Vol. 182, № 2, 1968.
19. Zhukov, M. Iu., On the upper critical load of thin nonshallow strictly convex shells of revolution. Izv. Sev.-Kavkaz. Nauch. Tsentra Vyssh. Shkoly. Estestv. Nauk, № 1, 1976.
20. Zhukov, M. Iu., Larchenko, V. V., Mel'nik, V. V., Srubshchik, L. S. and Tsariuk, L. B., Edge effects and buckling of thin nonshallow spherical shells. Transactions, Ninth All-Union Conference on Shells and Plates, pp. 119-21, Sudostroenie, Leningrad, 1975.

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